IRREGULAR SAMPLING IN APPROXIMATION SUBSPACES

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ABSTRACT

We describe the irregular sampling problem for discrete-time finite signals belonging to Fourier and wavelet based linear and nonlinear approximation subspaces. The problem is expressed as a linear system of equations which can be solved directly or iteratively. The existence of the solution depends on the rank of the matrix associated to the system. The standard iterative algorithm for reconstructing signals in Fourier subspaces is known as the Papoulis Gerchberg algorithm. Nonlinear approximation of a signal regardless of the subspace results in a smaller approximation error than a linear approximation. This is the motivation for developing the PG algorithm in nonlinear approximation subspaces. This variant makes use of the information obtained from the nonlinear approximation of the signal. We compare the reconstruction speeds of the PG algorithm on linearly and nonlinearly approximated signals in both Fourier and wavelet subspaces.

1. INTRODUCTION

In a communication channel data may be lost or corrupted: the receiving end obtains only an incomplete set of data packets and recovering the lost or corrupted packets can be cast as an irregular sampling problem. Two questions arise: how is an irregularly sampled signal reconstructed? and under what conditions may it be recovered with minimum error? These conditions depend on the space the signal belongs to. Fast reconstruction methods for recovering band-limited signals from an irregular set of samples are described in [2, 3, 8]. Irregular sampling theory in wavelets subspaces has also been studied [1, 10]. In this paper, we consider irregular sampling of signals belonging to Fourier and wavelet subspaces. In particular, signals that are linearly and nonlinearly approximated. The originality of our approach is that in the reconstruction we exploit the information given by these approximations.

2. IRREGULAR SAMPLING

In this section we express the irregular sampling problem for discrete-time signals with finite length as a linear system of equations. We will show how to solve the problem using a direct method involving the generalized inverse of the matrix. The problem may be also solved iteratively by the projection onto convex sets (POCS) method,[6]. The Papoulis Gerchberg (PG) algorithm is a standard iterative method for solving the irregular sampling problem for signals lying in Fourier subspaces. A faster and more efficient iterative method than the PG algorithm is the ACT algorithm for band-limited signals, [2, 3, 8] developed by NuHAG [1].
Next we introduce the direct method for the general irregular sampling problem, followed by the PG algorithm for band-limited signals.

2.1. Direct Method

Suppose we want to reconstruct a discrete-time signal $\mathbf{x} = \{x(n)\}_{n\in\mathcal{N}}, \mathcal{N} = \{0, 1, \ldots, N-1\}$, from a set of irregularly spaced samples $\{\mathbf{x}(n)\}_{n\in\mathcal{N}_K}$ where $\mathcal{N}_K = \{n_i\}_{i=1}^{K}$, $K < N$ and $0 \leq n_1 < \ldots < n_K \leq N-1$. If the signal belongs to a subspace spanned by M, $M \leq K < N$, basis vectors

$$\{\mathbf{g}_m = (g(0,m), g(1,m), \ldots, g(N-1,m)\}_{m\in\mathcal{M}}, \quad (1)$$

with $\mathcal{M} \subset \mathcal{N}$ and $|\mathcal{M}| = M$, then, $\forall n \in \mathcal{N}$,

$$x(n) = \sum_{m\in\mathcal{M}} c(m) g(n,m). \quad (2)$$

Since we only know $x(n)$ for $n \in \mathcal{N}_K$, the irregular sampling problem involves solving a system of K equations with M unknown variables $c(m), m \in \mathcal{M}$

$$\mathbf{G}_{\mathcal{N}_K\mathcal{M}}\mathbf{c}_{\mathcal{M}} = \mathbf{x}_{\mathcal{N}_K} \quad (3)$$

where $\mathbf{G}_{\mathcal{N}_K\mathcal{M}} = \{g(n,m)\}_{n\in\mathcal{N}_K, m\in\mathcal{M}}$, $\mathbf{c}_{\mathcal{M}} = \{c(m)\}_{m\in\mathcal{M}}$ and $\mathbf{x}_{\mathcal{N}_K} = \{x(n)\}_{n\in\mathcal{N}_K}$. If the rank of $\mathbf{G}_{\mathcal{N}_K\mathcal{M}}$ is equal

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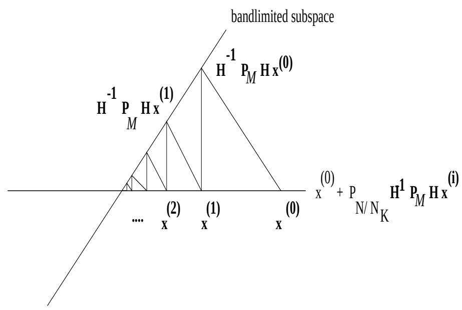

Figure 1: PG for M-BL signals,$\mathbf{H} = \mathbf{DFT}$.

to M then there exists a solution in the least squares sense obtained via the generalized inverse:

$$\mathbf{c}_{\mathcal{M}} = (\mathbf{G}^*_{\mathcal{N}_K\mathcal{M}}\mathbf{G}_{\mathcal{N}_K\mathcal{M}})^{-1}\mathbf{G}^*_{\mathcal{N}_K\mathcal{M}}\mathbf{x}_{\mathcal{N}_K}. \quad (4)$$

2.2. Iterative Method: Papoulis Gerchberg

The PG algorithm is a special case of the POCS method in that the convex sets are linear and was first defined for M band-limited signals: a signal is M band-limited if the M first components in the spectrum are nonzero and last $N - M$ are zero. It assumes that the signal belongs to two convex sets with non empty intersection. It involves two projections: the first projection is onto a band-limited subspace and the second projection is onto the space of unknown samples. This is exemplified in Figure 1.

Algorithm 1 *PG algorithm for M-BL signals*

Init $\mathbf{x}^{(0)} = \mathcal{P}_{\mathcal{N}_K}\mathbf{x}$

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(0)} + \underbrace{\mathcal{P}_{\mathcal{N}\setminus\mathcal{N}_K}}_{Proj_2}\underbrace{\mathbf{H}^{-1}\mathcal{P}_{\mathcal{M}}\mathbf{H}}_{Proj_1}\mathbf{x}^{(i)}$$

where $\mathbf{H}$ *is the* $\mathbf{DFT}$ *matrix,* $\mathcal{M} = \{0, 1, \ldots, M-1\}$*, and the projection operator is defined by*

$$\mathcal{P}_{\mathcal{S}}\mathbf{y} \;=\; \begin{cases} y(n) & \text{if } n \in \mathcal{S} \\ 0 & \text{if } n \in \mathcal{N}\setminus\mathcal{S} \end{cases} \quad (5)$$

with $\mathbf{y} = \{y(n)\}_{n\in\mathcal{N}}$.

$\mathbf{DFT}^{-1}\mathcal{P}_{\mathcal{M}}\mathbf{DFT}\mathbf{x}$ *is the projection of* $\mathbf{x}$ *onto the* M*-BL subspace.*

3. SIGNAL APPROXIMATION

The goal in signal approximation is to determine the basis which will best approximate a given signal. Consider for instance a signal $\mathbf{x} \in \ell^2(\mathcal{N})$. Given an orthogonal basis $\{\mathbf{g}_m\}_{m\in\mathcal{N}}$ for $\ell^2(\mathcal{N})$, we can write

$$\mathbf{x} = \sum_{m\in\mathcal{N}} < \mathbf{x}, \mathbf{g}_m > \mathbf{g}_m. \quad (6)$$

We can approximate $\mathbf{x}$ as a linear combination of an arbitrary set of M basis vectors. The basis vectors may be a fixed set or an adaptive set. We distinguish two types of approximations: linear and nonlinear.

3.1. Linear approximation

A linear approximation of $\mathbf{x}$ is a projection of $\mathbf{x}$ onto a subspace spanned by M basis vectors

$$\tilde{\mathbf{x}}_L = \sum_{m=0}^{M-1} < \mathbf{x}, \mathbf{g}_m > \mathbf{g}_m \quad (7)$$

with error

$$\tilde{e}_L = \|\mathbf{x} - \tilde{\mathbf{x}}_L\|^2 = \sum_{m=M}^{N} | < \mathbf{x}, \mathbf{g}_m > |^2. \quad (8)$$

Th basis set is fixed for all signals.

3.2. Nonlinear Approximation

Nonlinear approximation differs from linear approximation in that the set of basis vectors is not fixed but depends on the signal. The M basis vectors are chosen such that the approximation error is minimized. This set corresponds to basis vectors associated to the largest expansion coefficients $| < \mathbf{x}, \mathbf{g}_m > |$. We define this index set by

$$I_M = \{m_i \in \mathcal{N}, i = 1, \ldots, M : m_i = arg\max_{m\in\mathcal{N}} | < \mathbf{x}, \mathbf{g}_m > |\}.$$

Clearly I_M depends on the signal. A non-linear approximation of $\mathbf{x}$ is

$$\tilde{\mathbf{x}}_{NL} = \sum_{m\in I_M} < \mathbf{x}, \mathbf{g}_m > \mathbf{g}_m \quad (9)$$

whose approximation error is

$$\tilde{e}_{NL} = \|\mathbf{x} - \tilde{\mathbf{x}}_{NL}\|^2 = \sum_{m\notin I_M} | < \mathbf{x}, \mathbf{g}_m > |^2. \quad (10)$$

Non-linear approximation is never worse than linear approximation, since by definition

$$\tilde{e}_{NL} \le \tilde{e}_L. \quad (11)$$

4. RECONSTRUCTION IN APPROXIMATION SUBSPACES

The general superiority of the nonlinear approximation method suggests extending the PG algorithm to nonlinear approximation subspaces. We want to exploit the information of the nonlinear approximation in the reconstruction of the signal. For simulation purposes we suppose that the set I_M is

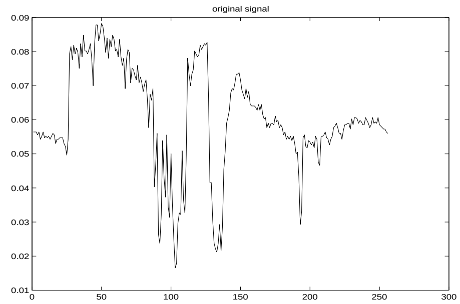

Figure 2: x:middle column of Lena, N=256.

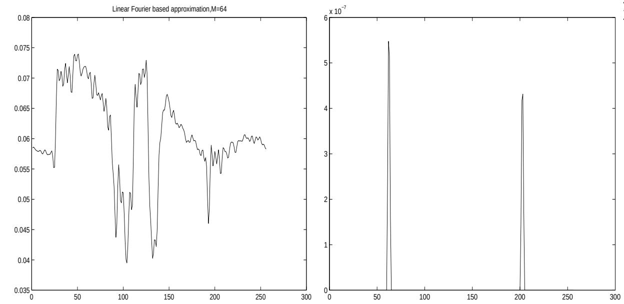

Figure 3: Fourier based linear approximation of signal: $\tilde{\mathbf{x}}_{LF} = \mathbf{DFT}^{-1}\mathcal{P}_{\mathcal{M}}\mathbf{DFTx}, \mathcal{M} = \{0, 1, \ldots, M-1\}, M = 64$; Difference between original and reconstruction,10^{-7}.

known. Next we consider the irregular sampling problem in four signal subspaces: Fourier based linear and nonlinear approximation subspaces and wavelet based linear and nonlinear approximation subspaces. Throughout this section we illustrate each case with an experimental example and a toy example. The experimental example shown in Figure 2 is a signal of length $N = 256$ and is the center column of the $N \times N$ Lena image. The purpose of the toy example is to help understand the algebra of the problem.

4.1. Fourier subspace

For signals belonging to Fourier subspaces in terms of the notation introduced in the direct method the matrix $\mathbf{G}$ is the inverse of the Discrete Fourier Transform ($\mathbf{IDFT}$) matrix. The expansion coefficients are simply the Fourier spectrum values of the signal.

$$\{g(n,m) = W_N^{nm} = e^{i2\pi/Nnm}\}_{n\in\mathcal{N}, m\in\mathcal{N}}. \qquad (12)$$

Consider the following example.

Example 4.1 *Suppose we want to reconstruct a discrete-time signal* $\mathbf{x}$ *of length* $N = 8$

$$\mathbf{x} = \mathbf{DFT}^{-1}\mathbf{X} \qquad (13)$$

, from $K = 4$ *irregularly spaced samples* $\mathbf{x}_{\mathcal{N}_K}$ *whose index set is* $\mathcal{N}_K = \{0, 1, 6, 7\}$*, where*

$$\mathbf{DFT}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & 1 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^3 & W_8^6 & W_8 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ 1 & W_8^4 & 1 & W_8^4 & 1 & W_8^4 & 1 & W_8^4 \\ 1 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8 & W_8^6 & W_8^3 \\ 1 & W_8^6 & W_8^4 & W_8^2 & 1 & W_8^6 & W_8^4 & W_8^2 \\ 1 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8 \end{bmatrix}$$

and $\mathbf{X}$ *are the Fourier components of the signal.*

4.1.1. *Linear approximation in Fourier subspaces*

Suppose $\tilde{\mathbf{x}}_{LF}$ is a linear approximation of $\mathbf{x}$ in Fourier subspace. Then $\tilde{\mathbf{x}}_{LF}$ is $M-BL$ and may be reconstructed using Algorithm 1. In terms of the example, let $M = 3$, then $\mathcal{M} = \{0, 1, 2\}$ then the problem in matrix form is

$$\begin{pmatrix} x(0) \\ x(1) \\ x(6) \\ x(7) \end{pmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_8 & W_8^2 \\ 1 & W_8^6 & W_8^4 \\ 1 & W_8^7 & W_8^6 \end{bmatrix}\begin{pmatrix} X(0) \\ X(1) \\ X(2) \end{pmatrix} \qquad (14)$$

The submatrix $\mathbf{G}_{\mathcal{N}_K\mathcal{M}}$ associated to the system of equations is a Vandermonde matrix [4]. Since all of the columns of a Vandermonde matrix are linearly independent, we have that the $rank(\mathbf{G}_{\mathcal{N}_K\mathcal{M}}) = M$ and therefore convergence of PG in this case is guaranteed.

4.1.2. *Nonlinear approximation in Fourier subspaces*

Suppose we want to reconstruct a Fourier based nonlinear approximation, $\tilde{\mathbf{x}}_{NLF}$ of a signal $\mathbf{x}$. The nonlinear approximation is characterized by having M, not contiguous but sparse nonzero values in the spectrum given by

$$\mathcal{I}_M = \{m_i \in \mathcal{N}, i = 1, \ldots, M : m_i = arg \max_{m\in\mathcal{N}} |X(m)|\}.$$

We substitute the set $\mathcal{M}$ in the PG algorithm by $\mathcal{I}_M$. The convergence of this variant is not guaranteed since the rank of the submatrix $\mathbf{G}_{\mathcal{N}_K\mathcal{I}_M}$ is not necessarily equal to M. For example if $\mathcal{N}_K = \{0, 1, 4, 5\}$ and $\mathcal{I}_M = \{0, 2, 4\}$ then $\mathbf{G}_{\mathcal{N}_K\mathcal{I}_M} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_8^2 & W_8^4 \\ 1 & 1 & 1 \\ 1 & W_8^2 & W_8^4 \end{bmatrix}$ is of rank two $< M = 3$. The system is not consistent and the PG algorithm will not converge to desired signal.

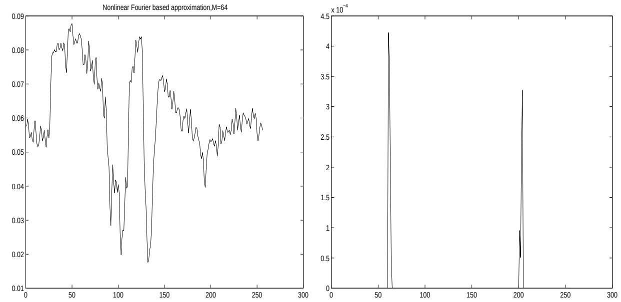

Figure 4: Fourier based nonlinear approximation of signal: $\tilde{\mathbf{x}}_{NLF} = \mathbf{DFT}^{-1}\mathcal{P}_{\mathcal{I}_M}\mathbf{DFTx}, \mathcal{I}_M = \{m_i \in \mathcal{N}, i = 1,\ldots,M : m_i = arg\max_{m\in\mathcal{N}} |X(m)|\}, M = 64$;Difference between original and reconstruction,10^{-4}.

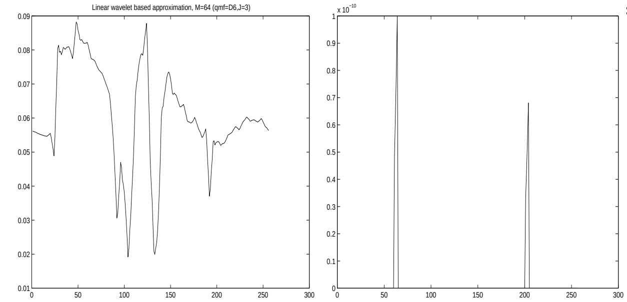

Figure 5: Wavelet based linear approximation of signal: $\tilde{\mathbf{x}}_{LW} = \mathbf{DWT}^{-1}\mathcal{P}_{\mathcal{M}}\mathbf{DWTx}, \mathcal{M} = \{0,1,\ldots,M-1\}, M = 64$,qmf=Daubechies length 6; Difference between original and reconstruction,10^{-10}.

4.2. Wavelet subspace

For signals belonging to wavelet subspaces, the transform matrix becomes the discrete wavelet transform, $\mathbf{DWT}$. The $\mathbf{DWT}$ depends on the quadrature mirror filters (qmf) used in the decomposition and the number of decomposition levels, J. We proceed with an example with similar parameters as in example 4.1 .

Example 4.2 *Suppose* $N = 2^3 = 8$,$J = 3$,$K = 4$, $\mathcal{N}_K = \{0,1,6,7\}$ *and that the sought signal lies in a subspace spanned by the Haar wavelet. The qmf filters are*

$$\frac{1}{\sqrt{2}}[1,1] \textit{ and } \frac{1}{\sqrt{2}}[1,-1].$$

Hence the signal is

$$\mathbf{x} = \mathbf{DWT}^{-1}(s_{30}, w_{30}, w_{20}, w_{21}, w_{10}, w_{11}, w_{12}, w_{13})^T$$

where

$$\mathbf{DWT}^{-1} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$s_{Jk}, w_{jk}, j = J,\ldots,0$ *and* $0 \le k \le J - j$ *are the scaling and wavelet coefficients, respectively.*

We refer the reader to [5, 7, 9] for the derivation of the discrete wavelet transform.

4.2.1. Linear approximation in wavelet subspaces

In [10] the Papoulis Gerchberg algorithm is generalized to signals lying in wavelet subspaces.They consider scale-time limited signals, which in our notation corresponds to saying that the M first coefficients of the wavelet transform are nonzero. The PG reconstruction algorithm for wavelet based linearly approximated signals is similar to Algorithm 1 except that the transform matrix is $\mathbf{H} = \mathbf{DWT}$. Since the $\mathbf{DWT}$ depends on the quadrature mirror filters the convergence of the PG algorithm is not as straightforward as in the linear approximation Fourier case. Returning to our example with $M = 3, \mathcal{M} = \{0,1,2\}$, the system of equations is

$$\begin{pmatrix} x(0) \\ x(1) \\ x(6) \\ x(7) \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{pmatrix} s_{30} \\ w_{30} \\ w_{20} \end{pmatrix}. \quad (15)$$

Since the rank of the matrix in equation (15) is equal to two $< M = 3$ the system is not consistent. The toy example demonstrates that even if the signal belongs to a subspace the choice of the quadrature mirror filters and their lengths are very important in the convergence of the PG algorithm.

4.2.2. Nonlinear approximation in wavelet subspace

Here we consider signals that belong to nonlinear approximation wavelet based subspaces. Similar to the nonlinear approximation Fourier based case, we substitute the index set $\mathcal{M}$ in Algorithm 1 by $\mathcal{I}_M$. This is the set of indices corresponding to the largest scale $|s_{Jk}|, 0 \le k \le J - j$ and wavelet $|w_{jk}|, j = J,\ldots,1$ coefficients. Suppose $\mathcal{N}_K = \{0,1,6,7\}$ and $\mathcal{I}_M = \{0,1,7\}$ then the problem to be solved is

$$\begin{pmatrix} x(0) \\ x(1) \\ x(6) \\ x(7) \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{pmatrix} s_{30} \\ w_{30} \\ w_{13} \end{pmatrix} (16)$$

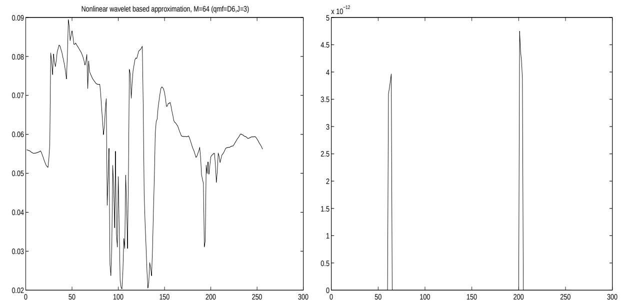

Figure 6: Wavelet based nonlinear approximation of signal: $\tilde{\mathbf{x}}_{NLw} = \mathbf{DWT}^{-1}\mathcal{P}_{\mathcal{I}_M}\mathbf{DWTx}, \mathcal{I}_M = \{m_i \in \mathcal{N}, i = 1,\ldots,M : m_i = arg\max_{m\in\mathcal{N}} |s_m|, |w_m|\}, M = 64$, qmf=Daubechies length 6; Difference between original and reconstruction,10^{-12}.

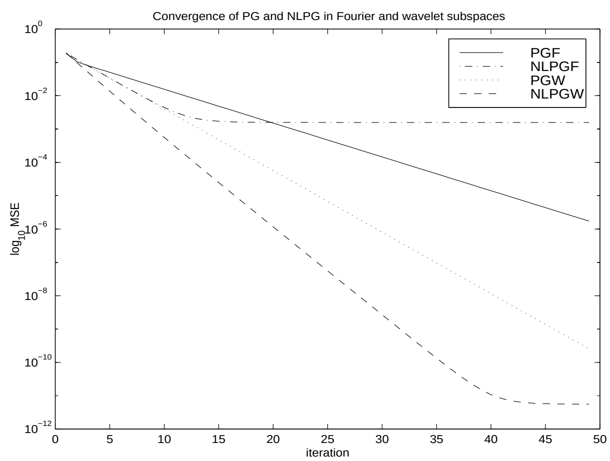

Figure 7: Convergence of PG on linear and nonlinear, Fourier and wavelet based signal approximation, $N = 256, K = 248, M = 64$.

This matrix is rank deficient. Again the convergence of the PG variant for signals in nonlinear approximation wavelet based subspaces depends on the structure and rank of the submatrix which in turn depends on the qmf's used in the decomposition.

4.3. Numerical Tests

Figure 7 summarizes the results of the numerical example shown in Figures 3 through 6. We compare the speed of reconstruction between linear and non-linear approximations in both Fourier and wavelet subspaces. We notice that after the 15th iteration the reconstruction error of the Fourier based nonlinearly approximated signal is not less than the linearly approximated signal which contradicts equation (11).

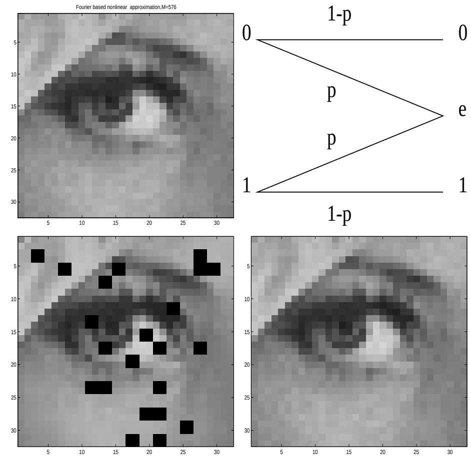

Figure 8: From top left corner: Fourier-based nonlinear approximation of Lena's eye $M = 24 \times 24$; Binary Erasure Channel, p=Prob(erasure)=0.10; Lena's eye with lost packets size 2×2; PG reconstruction of Fourier based nonlinear approximation (relative error=0.0009).

This hints that the associated system of equations is rank deficient and the PG algorithm will not converge to the desired signal. The wavelet case agrees with the theory. Hence if the system is not rank deficient then recovering the signal in a nonlinear approximation subspace gives a better reconstruction than the linear approximated signal. Next we redo the same experiment on images.

5. APPLICATIONS

In the introduction, we mentioned packet losses due to an erasure channel. Suppose that an image is sent through a channel whose erasure probability is p. We then apply the respective PG variants to the four approximations and compare the results. In Figure 8 a Fourier based nonlinear approximation of Lena's eye is sent through a binary channel with $p = 10\%$ and then reconstructed using PG in Fourier based nonlinear approximation subspaces. Which approximation gives the best reconstruction? Figure 9 shows that PG reconstruction error for the nonlinearly approximated image in each Fourier and wavelet subspace is less than the linear approximation which agrees with equation (11). It also shows that in this particular case Fourier basis are better than wavelet basis. The advantage of recovering signals in Fourier subspaces is that the information about the signal is spread throughout the whole of the Fourier spectrum. Wavelets are well known for their localization property which in this case is a drawback if the signal has contiguous missing samples or lost packets. On the other hand, if the signal is piecewise smooth then wavelet bases give a

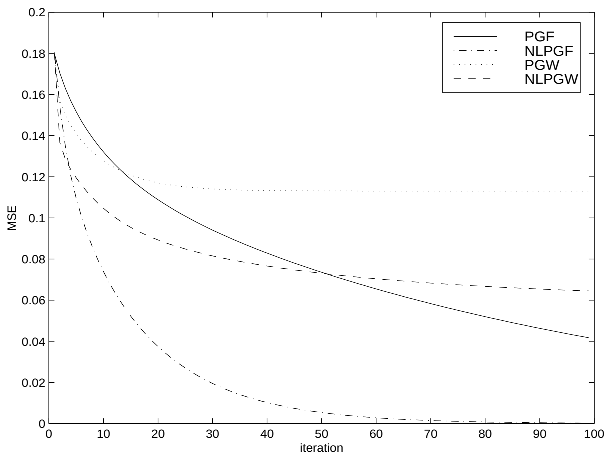

Figure 9: PG convergence of Fourier and wavelet based linear and nonlinear approximation of Lena's eye.

better approximation,[5, 7].

6. CONCLUSIONS AND FUTURE WORK

We described the irregular sampling problem for discrete time signals in Fourier and wavelet subspaces. We extended the PG algorithm to nonlinearly approximated signals lying in Fourier and wavelet subspaces. The advantage of recovering signals in nonlinear over linear approximation subspaces is that the recovery error is less. The disadvantage is that more information is needed to reconstruct, but this is a compression issue which we do not address here. The convergence of the PG variants depend on the rank of the associated submatrices: $\mathbf{DFT}$ in Fourier subspaces and $\mathbf{DWT}$ in wavelet subspaces. As future work, we are interested in shedding more light on the structure of these matrices. This in turn might enable us to find conditions which assures convergence of the PG algorithm for nonlinearly approximated signals.

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